Vibrational Aspects of the SU(2) Skyrmion

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Abstract

We treat the Skyrme model with the breathing mode in a situation involving two quartic terms. It is seen that there is a new limit for large e due to the breathing mode not found in the usual rotating hedgehog.

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1 Introduction

Recently many discussions have appeared in the literature [1 -4] about the stability of a chiral soliton without the Skyrme term, in particular the soliton solution to the nonlinear sigma model by means of the so called breathing mode (BM) [1, 2]. In this kind of solution, the radial variable, present in the hedgehog solution, is scaled by a dynamical variable so that its subsequent quantization furnishes an extra energy for the system. The proposal is that this energy is responsible for stability in a mechanism similar to the hydrogen atom. In fact, because of the BM, one can see in a rough approximation, using the uncertainty principle, that the classical energy (and radius), even when the rotations are absent, has a minimum. Although the results are good and comparable to that of the Skyrme model [5, 6] (which are obtained when as well as the nonlinear sigma term, the Skyrme term is also present) there are some problems that remain, one of them being the choice of the profile function $F(r)$ (determined in the Skyrme model by the minimization of the static mass). When the Skyrme term is absent, the static mass has no minimum and the profile function is quasi-arbitrarily chosen. This fact can lead to a new kind of instability as one can choose a function that results in divergent integrals even when the required boundary conditions are obeyed [7], so that the presence or not of the BM stabilizer term would make no difference.

As the BM has shown its importance in the nonlinear sigma model through the existence of a special stable soliton configuration, one can argue what is its relevance in the Skyrme model (for example, how accurately one can describe the nucleon and delta resonances). Although there are already some treatments of this subject in the literature [8, 9], mainly about the stability [10] of the Skyrme model with a vibrational solution, here we intend to explore the relationship between the Skyrme model and the nonlinear sigma model, more specifically how one can get the results of the nonlinear sigma model with BM as was done, for example, by Carlson [2], from the resolution of the Schrödinger equation of the Skyrme model with BM. In other words, we depart from the Skyrme model, which has a well defined profile function, perform the quantization of the rotational and vibrational degrees of freedom and take a suitable limit on the subsequent Schrödinger equation. As a result, we get the vibrating nonlinear sigma model with finite integrals. This kind of limit is a property only of the vibrating skyrmions. The same

limit exists for all static properties. This is the main idea of the present contribution.

2 The breathing mode

The SU(2) Skyrme model in a version that includes the two independent fourth order terms (Skyrme and symmetric quartic terms (SQT)) and in the chiral symmetry limit is described by the following Lagrangian density ($\hbar = c = 1$):

$$\mathcal{L}_{sk} = -\frac{F_\pi^2}{16} Tr(L_\mu L^\mu) + \frac{1}{32e^2} Tr([L_\mu, L_\nu]^2) + \frac{\gamma}{8e^2}\{[Tr(L_\mu L^\mu)^2]\}, \quad (1)$$

with $L_\mu = U^\dagger \partial_\mu U$, F_π=186 MeV the pion decay constant, e and γ parameters to be adjusted and $U = e^{i\vec{\tau}\cdot\vec{\pi}(\vec{x},t)}$ the pion field. Skyrme introduced soliton solutions in his model through the field configuration called hedgehog. In this ansatz, the pion field has a radial direction and depends on a function $F(r)$ called the chiral angle. This function is determined by the minimization of the static mass associated with the hedgehog solution. Well defined states of spin and isospin are obtained through a rotation of the classical solution. The BM can be taken into account by means of a scaling on the radial variable r:

$$U = A(t) e^{i\vec{\tau}\cdot\hat{r}F(r,R(t))} A^\dagger(t), \quad (2)$$

where $R(t)$ is the dynamical variable that characterizes the soliton breathing mode and $A(t) \in$ SU(2) is the rotation matrix. The use of the solution (2) in the Lagrangian density (1) and an integration over all space leads to the following Lagrangian:

$$L = R^3 b\dot{a}_\mu \dot{a}^\mu + R(k - \gamma m)\dot{a}_\mu \dot{a}^\mu - Rc - \frac{h - \gamma l}{R} + \dot{R}^2(f - \gamma n) + R\dot{R}^2 d, \quad (3)$$

with a_μ the four variables that parametrize the rotation matrix $A(t)$ and the constants are listed in appendix. We stress that the integrals c, b and d are the contributions from the nonlinear sigma term to the static mass, to the rotational term and to the BM, respectively. In the same order, h, k and f are the contributions from the Skyrme term and l, m and n the contributions from the symmetric quartic term.

One can pass to the Hamiltonian formulation through a Legendre transformation, whose result is:

$$H = \frac{\pi^2}{4R(R^2b + k - \gamma m)} + \frac{p_R^2}{4(Rd + f - \gamma n)} + Rc + \frac{h - \gamma l}{R}, \tag{4}$$

with $\pi^2 = \pi_\mu \pi^\mu$ the square of the momentum conjugate to a_μ and p_R the momentum conjugate to R. Now, we will write the Hamiltonian (4) through a set of reduced variables obtained when we make use of the constraint on the rotational variables ($a_\mu a^\mu = 1$). We remind the reader that if we first make the reduction of variables at the classical level and perform the quantization after, or first perform the quantization and afterwards restrict adequately the vectors of the Hilbert space, we obtain the same result [11, 12]. Another way is to construct a Hamiltonian compatible with the constraint by means of Dirac's procedure for constrained systems [13]. In this case there appears a contribution to the zero point energy inversely proportional to the moment of inertia [12, 14]. This contribution will not be treated in the present work.

Using the reduced coordinates $q = (a^i, R)$ and $p = (\pi_i, p_R)$, i=1,2,3, we write the Hamiltonian as:

$$H = \frac{1}{2} p_a g^{ab} p_b + cq^4 + \frac{h - \gamma l}{q^4}, \tag{5}$$

where:

$$g^{ab} = \begin{bmatrix} \dfrac{g^{ij}}{R(R^2b + k - \gamma m)} & 0 \\ 0 & \dfrac{1}{2(Rd + f - \gamma n)} \end{bmatrix} \tag{6}$$

and

$$g^{ij} = \frac{1}{2}(\delta^{ij} - q^i q^j). \tag{7}$$

We write the quantum Hamiltonian via the DeWitt [15] prescription:

$$\begin{aligned} H\psi &= -\frac{1}{2g^{1/2}} \partial_a (g^{1/2} g^{ab} \partial_b \psi) + \left(q^4 c\psi + \frac{h - \gamma l}{q^4} \psi \right) \\ &= -\frac{1}{2g^{1/2}} \partial_i (g^{1/2} g^{ij} \partial_j \psi) - \frac{1}{2g^{1/2}} \partial_4 (g^{1/2} g^{44} \partial_4 \psi) + Rc\psi + \frac{h - \gamma l}{R} \psi \end{aligned} \tag{8}$$

or

$$-\frac{1}{4d}\left\{\frac{\partial^2}{\partial R^2}+\frac{4R^4+(5(f-\gamma n)/d+(k-\gamma m)/b)R^2+2(f-\gamma n)(k-\gamma m)/bd}{R(R^2+(f-\gamma n)/d)(R^2+(k-\gamma m)/b)}\frac{\partial}{\partial R}\right\}\psi$$
$$+\frac{R^2+(f-\gamma n)/d}{R}\left(\frac{j(j+1)}{R(bR^2+k-\gamma m)}+Rc+\frac{h-\gamma l}{R}\right)\psi=\frac{R^2+(f-\gamma n)/d}{R}E\psi, \quad (9)$$

where g is the determinant of the metric tensor. We make a transformation on the wave function ψ as indicated in [9],

$$\psi=\left\{\frac{2(dR^2+f-\gamma n)}{R}\right\}^{1/4}\left\{2R(bR^2+k-\gamma m)\right\}^{-3/4}\phi. \quad (10)$$

and eliminate the first derivative from the equation. So:

$$\begin{aligned}
&-\frac{1}{4d}\frac{\partial^2}{\partial R^2}\phi\\
&+\frac{8R^6+(18(f-\gamma n)/d+10(k-\gamma m)/b)R^4}{16d\left((f-\gamma n)/d+R^2\right)^2\left((k-\gamma m)/b+R^2\right)^2}\phi\\
&+\frac{(15(f-\gamma n)^2/d^2+18(f-\gamma n)(k-\gamma m)/db)R^2}{16d\left((f-\gamma n)/d+R^2\right)^2\left((k-\gamma m)/b+R^2\right)^2}\phi\\
&-\frac{(k-\gamma m)^2R^2/b^2+18(k-\gamma m)(f-\gamma n)^2/bd^2-6(f-\gamma n)(k-\gamma m)^2/db^2}{16d\left((f-\gamma n)/d+R^2\right)^2\left((k-\gamma m)/b+R^2\right)^2}\phi\\
&+\frac{(R^2+(f-\gamma n)/d)}{R}\left(\frac{j(j+1)}{R(bR^2+k-\gamma m)}+cR+\frac{h-\gamma l}{R}\right)\phi\\
&=\frac{R^2+(f-\gamma n)/d}{R}E\phi.
\end{aligned} \quad (11)$$

In order to obtain a solution to this equation, we observe that the behavior of ϕ near the origin is:

$$\phi(R\to 0)\sim R^{\frac{1}{2}+\left\{\frac{1}{4}+4(f-\gamma n)\left(\frac{j(j+1)}{k-\gamma m}+h-\gamma l\right)\right\}^{1/2}} \quad (12)$$

and we require that ϕ goes to zero at infinity.

3 The limit of large e

Usually in the Skyrme model, the two free parameters of the theory are chosen to reproduce the nucleon and delta masses. As a result, the value of e that reproduces such masses does not correspond to the e value that minimizes them. This is not so important in the context of the value of e to be used, because there is no necessity to use the e that minimizes the mass. But it is important in the sense that it shows that for e going to zero or e going to infinity, the mass diverges (see figure 1). Now, the question is what happens with the mass in that limit when along with the rotations, the BM is also present? The answer is that for small e the mass still diverges, but for sufficiently large e, the mass becomes a constant in e and all works as if only the nonlinear sigma term were present, with a profile function F that minimizes the static mass of the Skyrmion.

The results for the nucleon and delta masses and their excitations are the same as those obtained some time ago by Carlson [2]. In that work, Carlson pointed out that it is possible to obtain stable soliton configurations from the nonlinear sigma model when the Skyrme term is used as a constraint in the action of the system. This was shown using scale invariant arguments in the functional integral for the field configurations U, dividing the configurations into equivalence classes and extending the functional integral to the equivalence classes of configurations. The resulting functional integral depends of a dilatated field and involves a restriction on a particular function $G(U)$ of U. This function must preserve the chiral limit and must not be invariant by dilatations as, in this case, it would not be possible to choose a field configuration that was representative of its class. Carlson chose for $G(U)$ the Skyrme term. With this choice, it is possible to solve the Euler-Lagrange equation without the explicit presence of the Skyrme term in the action. So, Carlson obtains a hamiltonian of the nonlinear sigma model with integrals dependent of the chiral angle from the Skyrme term.

Now, we will call y the dimensionless variable that scales the radial variable in the nonlinear sigma model. It is related to R by $R = ey$. If one makes this replacement in the Schrödinger equation (9) is possible to verify that for $e \to \infty$, it reduces to:

$$-\frac{1}{4yd}\left\{\frac{\partial^2}{\partial y^2}+\frac{4}{y}\frac{\partial}{\partial y}\right\}\psi+\frac{j(j+1)}{by^3}\psi+cy\psi=E\psi, \tag{13}$$

which is e independent. As the SQT and the Skyrme term have the same e dependence, the equation has the same functional form when γ is zero or not in the limit of large e, with the only difference being the function F that minimizes the classical mass. The expression (13) is the Schrödinger equation for the BM when only the nonlinear sigma term is present. In fact, the numerical resolution of equation (9) shown in figure (1), tells us that for $e \geq 80$ the mass is already that calculated by Carlson. The other static properties have the same kind of limit. As an example, we write down the expression for the axial coupling g_a. It is easy to verify that g_a, given in the vibrating Skyrme model by

$$\begin{aligned} g_a &= \frac{1}{e^2}\frac{\pi}{3}\frac{\int R^2 \mid \psi \mid^2 \sqrt{g} dR}{\int \mid \psi \mid^2 \sqrt{g} dR} \int_0^\infty d\rho \rho^2 \left(F' + \frac{sin(2F)}{\rho} \right) \\ &+ \frac{8\pi}{3e^2} \int_0^\infty d\rho sin^2 F F' \\ &+ \frac{4\pi}{3e^2} \left(\int_0^\infty d\rho\rho sin(2F)(F')^2 + \int_0^\infty d\rho sin^2 F \frac{sen(2F)}{\rho} \right), \end{aligned} \tag{14}$$

is reduced, in the limit of large e, to:

$$g_a = \frac{\pi}{3}\frac{\int y^2 \mid \psi \mid^2 \sqrt{g} dy}{\int \mid \psi \mid^2 \sqrt{g} dy} \int_0^\infty d\rho \rho^2 \left(F' + \frac{sin(2F)}{\rho} \right), \tag{15}$$

which is the corresponding expression in the nonlinear sigma model. The same holds for all other static properties.

4 Numerical results

Figure (1) shows the mass curve for the case of the rotating hedgehog with the Skyrme term (case I), Skyrme term plus BM (case II) and Skyrme and SQT terms plus BM (case III). The value used for the pion coupling constant is $F_\pi = 186$ MeV and γ is 0.11 which is the maximum γ value that mimimizes the classical mass. This value agrees with those found in the literature [16]. The discrepancy between cases II and III is due to the presence or absence of the SQT. As seen in equation (11), the SQT contributes negatively to the energy so that for low e, the mass calculated when the SQT is present is

lower than that calculated when only the Skyrme term is present. In the high e region, the only difference is in the constants b, c and d of equation (13) which are calculated with different profile functions. For case II they are $125.01/F_\pi e^3$, $18.23 F_\pi/e$ and $160.79/F_\pi e^3$ and for case III $52.57/F_\pi e^3$, $14.72 F_\pi/e$ and $72.87/F_\pi e^3$ respectively (see table in the appendix). The asymptotic mass value for case II is 1101 MeV and for case III 1152 MeV compared with an experimental value given by 939 MeV.

As the energy is not absolutely defined in our treatment, we decided to search for mass differences (fig. 2). Initially we have two free parameters in the model. One of them, as stated before, was chosen to fit the pion decay constant. The other, in principle, could be chosen to fit the mass diference between the states with j=1/2 and j=3/2. The problem is that for small e the mass difference is too small and for large e it is independent of e so that in this region we have only one free parameter. As we choose to fit F_π, the biggest mass difference for case II is 285 MeV and for case III is 306 MeV. The experimental value is 293 MeV. The differences between cases II and III arise because the centrifugal term in III is bigger than II implying that the mass difference is also bigger. In figure (2) we also plot the mass difference for the rotating hedgehog with the Skyrme term. As is well known, it behaves like e^3. Finally, to illustrate the behaviour of the other static properties, the isoscalar radius curve is plotted in figure (3). One can see that for large e, the isoscalar radius for the rotating hedghog goes to zero, as expected, because in that region the effect of the Skyrme term is weak and the soliton collapses. On the other side, when the BM is present, it is verified that for large e the isoscalar radius goes to a constant value. For case II, the mean charge radius is 0.56 fm and for case III is 0.58 fm. Another example is the isoscalar mean magnetic radius. For case II, it is 0.96 fm and for case III is 0.92 fm.

5 Conclusions

We have studied the Skyrme model in a solution for the pion field that includes both rotational and vibrational degrees of freedom. It was found that the presence of the breathing mode in the solution allows one to take the limit $e \to \infty$ - a situation not found in the usual rotating hedgehog. As a consequence, we recovered the earlier results of Carlson [2]. We did this study in two situations: one involving only the Skyrme term and the other involving

the Skyrme term plus the symmetric quartic term. The limit for large e is the same for the two cases with the only difference being the function F that minimizes the classical mass. Some static properties were presented and it was noticed that the masses of the nucleon and delta states as well as the mass difference between them are larger when the symmetric quartic term is present. This is due to centrifugal effects. We also think that the method presented here provides some justification for the use of a particular chiral angle in the model proposed by P. Jain et al. [1].

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Appendix: Integrals containing the chiral angle

Let $F(\rho)$ be the chiral angle, $s = sinF$, $c = cosF$, F_π the pion decay constant $(\pi \rightarrow \mu + \nu)$ and e the Skyrme dimensionless parameter. So, the explicit form for the constants appearing in the Lagrangean is given by:

$$b = \frac{4\pi}{3e^3 F_\pi} \int_0^\infty s^2 \rho^2 d\rho, \tag{1}$$

$$c = \frac{\pi F_\pi}{2e} \int_0^\infty \left(\left(\frac{\partial F}{\partial \rho} \right)^2 + \frac{2s^2}{\rho^2} \right) \rho^2 d\rho, \tag{2}$$

$$d = \frac{\pi}{2e^3 F_\pi} \int_0^\infty \left(\frac{\partial F}{\partial \rho} \right)^2 \rho^4 d\rho, \tag{3}$$

$$f = \frac{4\pi}{e^3 F_\pi} \int_0^\infty s^2 \left(\frac{\partial F}{\partial \rho} \right)^2 \rho^2 d\rho, \tag{4}$$

$$g = \frac{16\pi}{3e^3 F_\pi} \int_0^\infty s^2 \left(\left(\frac{\partial F}{\partial \rho} \right)^2 + \frac{s^2}{\rho^2} \right) \rho^2 d\rho, \tag{5}$$

Integrals	$\gamma = 0$	$\gamma = 0.11$
b	125.0145	52.573
c	18.23082	14.716
d	160.7893	72.874
f	43.91	39.994
g	88.52515	75.37
h	18.23	22.965
l	-	74.988
m	-	194.834
n	-	193.077

Table 1: Numerical values for the integrals containing the chiral angle when only the Skyrme term is present (γ=0) and when the SQT (γ=0.11) is added. The parameters e and F_π are not included.

$$h = \frac{2\pi F_\pi}{e} \int_0^\infty s^2 \left(2 \left(\frac{\partial F}{\partial \rho} \right)^2 + \frac{s^2}{\rho^2} \right) d\rho, \tag{6}$$

$$l = \frac{2\pi F_\pi}{e} \int_0^\infty \left(\left(\frac{\partial F}{\partial \rho} \right)^2 + 2\frac{s^2}{\rho^2} \right)^2 \rho^2 d\rho, \tag{7}$$

$$m = \frac{32\pi}{3e^3 F_\pi} \int_0^\infty s^2 \left(\left(\frac{\partial F}{\partial \rho} \right)^2 + 2\frac{s^2}{\rho^2} \right) \rho^2 d\rho, \tag{8}$$

$$n = \frac{4\pi}{e^3 F_\pi} \int_0^\infty \left(\frac{\partial F}{\partial \rho} \right)^2 \left(\left(\frac{\partial F}{\partial \rho} \right)^2 + 2\frac{s^2}{\rho^2} \right) \rho^4 d\rho, \tag{9}$$

The coefficients c, h and l are the contributions for the soliton static mass from the nonlinear sigma term, the Skyrme term and SQT respectively. In the same way, the coefficients b, g and m are the contributions for the rotational energy and d, f and n the contributions for the vibrational energy.

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Figure 1: Energy curve for the rotating hedgehog with and without BM in function of the Skyrme parameter e. In cases II (dashed line) and III (continuous line), where the BM is present, the energy decreases with e until a region where it becomes e independent. In case I (dotted line), the energy diverges as e tends to zero or infinity.

Figure 2: Mass difference curve between the states j=1/2 and j=3/2 for the rotating hedgehog with and without BM in function of the Skyrme parameter e. In cases II (dashed line) and III (continuous line), the mass difference increases until a region where it becomes e independent. In case I (dotted line), the mass difference diverges as e goes to the infinity.

Figure 3: Isoscalar radius curve for the rotating hedgehog with and without breathing mode in function of the Skyrme parameter e. The notation is the same as in figure 1. In the case that the breathing mode is present, the isoscalar radius decreases with e until it reaches a minimum and tends to a constant. In the other case, the radius goes continuously to zero as the strength of the Skyrme term becomes weaker.